integration. For the $\pi^0$ and $\gamma$ reactions with the magnets set for an $\alpha$ momentum of 1060 MeV/$c$, the acceptances were evaluated as 0.79 sr-cm and 0.47 sr-cm, respectively. The term $M(x)$ does not take into account the multiple scattering in the $T1$ and $T2$ counters. An additional correction was made for this effect which, for the $\pi^0$ cross section (5), was 3%.

To get a feeling for these numbers we could arbitrarily assign numerical values of 0.13 sr to $\Delta\Omega_{\text{c.m.}}$, 0.55 to $M$, and 11 cm to $L$, where $L$ is an effective target thickness

$$L=\int_0^L dx. \tag{8}$$

The only meaning that should be attached to these three numbers is that their product equals the value given above (0.79 sr-cm) for the correct numerical evaluation of (7) for the $\pi^0$ reaction (1).

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# Theory of the $J=\frac{3}{2}$, $I=\frac{3}{2}$ $\pi N$ Resonance*

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We calculate the position $W_R$ and width $\gamma_{33}$ of the $J=\frac{3}{2}$, $I=\frac{3}{2}$ $P$-wave $\pi N$ resonance, using partial-wave dispersion relations. In the present calculation we treat as given the nucleon and $\rho$-meson masses and coupling constants, which determine the long-range part of the forces. The parameters, which characterize the distant part of the left-hand cut, are fixed by using the expressions for the $(\frac{3}{2},\frac{3}{2})$ $P$-wave $\pi N$ state given by fixed energy dispersion relations, in a region where they are valid without subtractions, in a way used by Balázs for the $\pi\pi$ problem. We then impose the self-consistency demand that the position and width of the $(\frac{3}{2},\frac{3}{2})$ resonance used as input values in the crossed channel in the fixed-energy dispersion relation be the same as the calculated values of the position and width. The preliminary results of the calculation are $W_R\approx m+2.35$ and $\gamma_{33}\approx 0.14$. The experimental values are $W_R=m+2.17$ and $\gamma_{33}\approx 0.12$, (where $m$ is the nucleon mass and we use units in which $\hbar=c=m_\pi=1$). These results constitute the first part of the intended self-consistent calculation of the nucleon mass and $(\frac{3}{2},\frac{3}{2})$ resonance position, exploiting the "reciprocal bootstrap" mechanism discussed by Chew.

WE present here some preliminary results on the first part of a program of self-consistent calculation of the nucleon mass and the position of the 3/2–3/2 resonance of the $\pi N$ system using unitarity and analyticity. The idea that is being exploited in this attempt is that the nucleon in the crossed channel provides the main force for the 3/2–3/2 resonance, and that the 3/2–3/2 resonance in the crossed channel provides the main force for the binding of the nucleon. The feasibility of such a "reciprocal bootstrap" mechanism has been discussed recently by Chew.[1]

The existence of this resonance and its width have been well understood on the basis of the Chew-Low theory[2] which brought out the dominant role played by the nucleon in the crossed channel. Its position, however, has defied all dispersion theoretical attempts so far, for want of a satisfactory method for taking into account the short-range effects whose importance was emphasized in the work of Frautschi and Walecka.[3] A new effective-range method has been developed recently by Balázs[4] for treating the distant part of the left-hand cut and successfully applied to the problems of $\pi\pi$ scattering[4] and the isovector part of the electromagnetic structure of the nucleon.[5] We show that it also leads to some very interesting results for the present problem.

In the present note we confine our discussion to the first part of the program, viz., determination of the position of the 3/2–3/2 resonance treating the nucleon in the crossed channel as a given fixed singularity. We use the $N/D$ method, and the amplitude we choose is the

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[1] G. F. Chew, Phys. Rev. Letters **9**, 233 (1962). See also F. Low, *ibid.* **9**, 277 (1962).

[2] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

[3] S. C. Frautschi and J. D. Walecka, Phys. Rev. **120**, 1486 (1960).

[4] L. Balaźs, Phys. Rev. **128**, 1939 (1962); **129**, 872 (1963).

[5] Virendra Singh and B. M. Udgaonkar, Phys. Rev. **128**, 1820 (1962).

same as that used by Frautschi and Walecka, viz., $g_{33}=(W^2/q^3)e^{i\delta_{33}}\sin\delta_{33}$. Unlike these authors, however, we work in the $s$ plane instead of the $W$ plane, thereby avoiding a cut along the imaginary axis.[6] We then use an effective-range approximation to treat the $N/D$ equations, wherein the distant part of the left-hand cut is replaced by a few (in fact, two) poles whose positions are fixed *a priori* so as to approximate the kernel in the equation for the numerator function sufficiently well in the region of interest.[7] We have a third pole at $s=m^2$ to take care of the short cut $(m-1/m)^2\leq s\leq m^2+2$. Its residue $b_0$ is, however known[2,3] to be $(8f^2m^3/3)D(m^2)$. Thus our numerator function reads[8]

$$N(s)=\sum_{i=0}^{2} b_i/(s-s_i), \tag{1}$$

with $s_0=m^2$, $s_1=-m^2$, $s_2=-16m^2$, and it involves two unknown residues $b_1$ and $b_2$. The denominator function is then given by

$$D(s)=1-\frac{s-(m-1)^2}{\pi}\int_{(m+1)^2}^{\infty} ds'\frac{q'^3/s'}{(s'-s)[s'-(m-1)^2]}\times\sum_{i=0}^{2}\frac{b_i}{s'-s_i}. \tag{2}$$

The residues $b_i$ are now determined by matching the values of the amplitude so written, and its derivative, at a suitably chosen point, with values calculated from the fixed $s$ dispersion relation

$$A^i(s,u,t)=\frac{R_s^i}{m^2-s}+\frac{R_u^i}{m^2-u}+\frac{1}{\pi}\int_{(m+1)^2}^{\infty}\frac{A_u^i(u',s)du'}{u'-u}+\frac{1}{\pi}\int_4^{\infty}\frac{A_t^i(t',s)dt'}{t'-t} \tag{3}$$

satisfied by the invariant amplitudes $A^i$. The matching point has to be chosen so as to satisfy two criteria: (1) The known Regge behavior of the amplitudes[9] must ensure that the partial-wave projection of the fixed $s$ dispersion relation gives a convergent expression at that point. (2) Since, in practice, we make a partial-wave expansion of the absorptive parts $A_u$, $A_t$ occurring in Eq. (3), the respective partial-wave expansions must be convergent. The fixed point we choose is $s=(m-1)^2$.

Our use of the fixed $s$ dispersion relation to evaluate $b_i$ differs in an important respect from that of previous authors.[10] The usual procedure is to make a partial-wave expansion of $A_u$ and $A_t$ and retain as many waves as possible. However, the only important angular momentum states in the $u$ and $t$ channels about which one has reliable information are the 3/2–3/2 state and the $J=1$, $T=1$ state. There are higher resonances in the $\pi N$ channel, and very probably in the $\pi\pi$ channel too; but since the information on them is not adequate (e.g., one does not even know for sure whether the 600-MeV bump in $\pi N$ scattering is a resonance or not), we have found it more convenient to use the known equivalence between a low-energy resonance in one channel and high-energy, high angular momentum states in the crossed channels.[11] More explicitly, we can write (omitting the suffix $i$ and the pole terms for the time being)

$$A(s,u,t)=\frac{1}{\pi}\int_{3/2-3/2}\frac{A_u(u',s)du'}{u'-u}+\frac{1}{\pi}\int_{\rho}\frac{A_t(t',s)}{t'-t}dt'+\frac{1}{\pi}\int_{\text{high }u}\frac{A_u(u',s)du'}{u'-u}+\frac{1}{\pi}\int_{\text{high }t}\frac{A_t(t',s)dt'}{t'-t},$$

where the first integral runs over the 3/2–3/2 resonance, the second over the $\rho$, and the last two integrals represent the remaining contributions from high energies. In these latter, if one expresses $A_u$ and $A_t$ in terms of the double spectral functions, one readily sees that

$$\frac{1}{\pi}\int_{\text{high }u}\frac{A_u(u',s)du'}{u'-u}+\frac{1}{\pi}\int_{\text{high }t}\frac{A_t(t',s)dt'}{t'-t}\simeq\frac{1}{\pi}\int_{3/2-3/2}\frac{A_s(s',t)}{s'-s}, \tag{4}$$

if one neglects the contributions of distant double spectral functions in the spirit of a strip approximation (see Fig. 1). One thus gets[12]

$$A^i(s,u,t)\simeq\frac{R_s^i}{m^2-s}+\frac{R_u^i}{m^2-u}+\int_{3/2-3/2}\frac{A_u^i(u',s)du'}{u'-u}+\frac{1}{\pi}\int_{\rho}\frac{A_t^i(t',s)dt'}{t'-t}+\frac{1}{\pi}\int_{3/2-3/2}\frac{A_s^i(s',t)ds'}{s'-s}. \tag{5}$$

We use Eq. (5) to project out $g_{33}$ and its derivative at $s=(m-1)^2$. We may then write

$$g_{33}=g_{33}^{(N)}+g_{33}^{(33)}+g_{33}^{(\rho)}+g_{33}^{(h)}. \tag{6}$$

Here $g_{33}^{(N)}$, $g_{33}^{(33)}$, and $g_{33}^{(\rho)}$ denote, respectively, the contributions of the nucleon pole, the 3/2–3/2 resonance, and the $\rho$, in the crossed channels. The expressions for

[6] The kinematic singularities occurring when one works in the $s$ plane present no difficulties in our method which does not involve the use of the discontinuities across the cuts.

[7] See references 4 and 5 for details regarding this method.

[8] In fixing the pole positions $s_1=-m^2$, $s_2=-16m^2$, we have neglected the part of the cut $0\leq s\leq(m-1)^2$, since it is known (reference 3) to be weak.

[9] Virendra Singh, University of California Radiation Laboratory Report, UCRL-10416, 1962 (unpublished).

[10] Cf., for example, Louis Balázs, reference 4.

[11] See, for example, G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **5**, 580 (1960).

[12] Equation (5) is very similar to the one used by J. Bowcock, W. N. Cottingham, and D. Lurié [Nuovo Cimento **16**, 918 (1960)]. These authors, however, use it for a different purpose. In fact, they use it as a sum rule in the physical region of the $\pi N$ channel, while we use it only at $s=(m-1)^2$, where it can be used with much greater confidence.

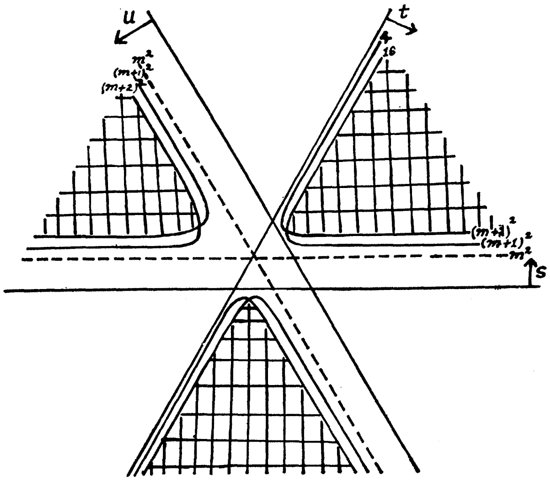


FIG. 1. The Mandelstam diagram for $\pi N$ scattering. The shaded area shows the regions of the double spectral functions which are neglected in the approximation of Eq. 5.

these have been written down by Frautschi and Walecka.[13] The expression for $g_{33}^{(\rho)}$ involves the parameters $\gamma_1$ and $\gamma_2$ which are to be evaluated from one's knowledge of the nucleon electromagnetic structure. We use the results of our recent investigation of this problem[5] which gives $\gamma_1=-4.91$ and $\gamma_2=-11.7$. Finally, $g_{33}^{(h)}$ denotes the high-energy contributions in the crossed channels, expressed in terms of the 3/2–3/2 resonance in the direct channel as discussed above. It is given by

$$g_{33}^{(h)}=\frac{\gamma_{33}W_R^2}{W_R-W}\frac{(W+m)^2-1}{(W_R+m)^2-1}, \qquad (7)$$

where $\gamma_{33}$ is the reduced half-width of the 3/2–3/2 resonance and $W_R$ its position.

The remaining procedure is quite straightforward. Equations (5), (6), and (7) (with some assumed values of $W_R$ and $\gamma_{33}$) are used to determine $b_i$, and then the denominator function $D$ is given by Eq. (2). The position where $D$ vanishes and its slope (together with the value of $N$) then yield, respectively, the position $W_R$ and the width $\gamma_{33}$ of the 3/2–3/2 resonance. One sees immediately that there is a self-consistency problem here: The input values used for $W_R$ and $\gamma_{33}$ in $g_{33}^{(33)}$ and $g_{33}^{(h)}$ in order to determine $b_i$ must be the same as the output values calculated from the $D$ function. One uses successive iterations, and it turns out that the results converge quite rapidly. This is due to the fact that if, for example, one takes too high an input value for $W_R$ in $g_{33}^{(33)}$ and $g_{33}^{(h)}$ (with a given $\gamma_{33}$), then one ends up with too low an output value for it, and vice versa. In this way, we finally get $W_R\simeq m+2.35$ and $\gamma_{33}\simeq 0.14$.

The position so obtained is in fairly good agreement with the experimental value[14] of $W_R=m+2.17$, though the width is somewhat larger[15] than the experimental value[14] of $\gamma_{33}\simeq\frac{4}{3}f^2\simeq 0.12$. It need hardly be emphasized that we have been able to get this agreement without introducing any free parameters (e.g., cutoffs) in the theory. The only external information that has been fed in is the knowledge of the nucleon form factors[16] which was used in evaluating $g_{33}^{(\rho)}$.

The result looks very promising for the complete self-consistent calculation which is now in progress, and which, we hope, will give both the nucleon mass and the 3/2–3/2 resonance in approximately the correct positions.


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[13] Equations (6.10), (7.13), and (7.35) of reference 3. There is, however, a misprint in Eq. (7.13), which should read

$$g_1^{\times}=\frac{W_R^2}{9q^4}\left(\frac{1}{4}\right)\left(\frac{f}{\mu}\right)^2 q_r^2\Big\{[(W+M)^2-\mu^2]\times\left[\frac{3x^{\times}(W_R+2M-W)}{(W_R+M)^2-\mu^2}+\frac{W_R-2M+W}{(W_R-M)^2-\mu^2}\right]\left(2-a\ln\frac{a+1}{a-1}\right)-[(W-M)^2-\mu^2]\left[\frac{3x^{\times}(W_R+2M+W)}{(W_R+M)^2-\mu^2}+\frac{W_R-2M-W}{(W_R-M)^2-\mu^2}\right]\times\left[-3a+\left(\frac{3a^2-1}{2}\right)\ln\frac{a+1}{a-1}\right]\Big\}.$$

We use this expression after replacing $4f^2/3$ by $\gamma_{33}$.

[14] We are taking the experimental values of $W_R$ and $\gamma_{33}$ from S. W. Barnes, B. Rose, G. Giacomelli, J. Ring, K. Miyake, and K. Kinsey, Phys. Rev. **117**, 226 (1960).

[15] It is very probable that the width will be improved by inclusion of inelastic effects in the equation for $D$. This seems to happen, for example, in the $\pi\pi$ problem [Louis Balázs (private communication)]. This effect is being investigated.

[16] It turns out that the result is rather insensitive to the contribution of $\rho$, e.g., changing $\gamma_1$ and $\gamma_2$ by a factor of 2 does not alter $W_R$ by more than 5%. On the other hand, high-energy contributions from $t$ and $u$ channels, included through the approximation of Eq. (4), are quite important. The main contribution here probably comes from the $T=0$, $J=2$ resonance at $\sim$1 BeV, which has been conjectured for some time [C. Lovelace, Nuovo Cimento **25**, 730 (1962); also G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1962)] and which now seems to have been confirmed experimentally [A. Baker, H. Brody, V. Hagopian, and W. Selove, Phys. Rev. Letters **9**, 221 (1962)]. This contribution is now being considered in more detail.